Lecture 18

Wednesday, October 13, 2021 12:26 AM

* Prayer

* Spiritual thought

* Forced vibrations:

$$my'' + \gamma y' + ky = F(t) = A\cos\omega t \qquad (*)$$

Over time, the object seems to vibrate at the same frequency as the force. Why so?

To solve for y from $(*)$, we have

$$y_c = \begin{cases} c_1 e^{-r_1 t} + c_2 e^{-r_2 t} \\ e^{-at}(c_1\cos\beta t + c_2 \sin\beta t) \end{cases} \xrightarrow{t\to\infty} 0 \quad (\text{assuming } \gamma > 0)$$

Particular solution:

$$y_p = B\cos\omega t + C\sin\omega t$$

$$y = \underbrace{y_c}_{\substack{\downarrow \\ 0 \\ \text{transient sol.}}} + y_p \overset{t\text{ large}}{\approx} y_p = B\cos\omega t + C\sin\omega t = \overbrace{D\cos(\omega t - \delta)}^{\text{steady-state sol.}}$$

$D\cos(\omega t-\delta)$: Same frequency as the force

The ω that causes D to be maximum is the ω causing resonance.

$$\omega^* = \omega_0\sqrt{1 - \frac{\gamma^2}{2mk}}, \text{ where } \omega_0 = \sqrt{\frac{k}{m}} \text{ is the natural frequency.}$$

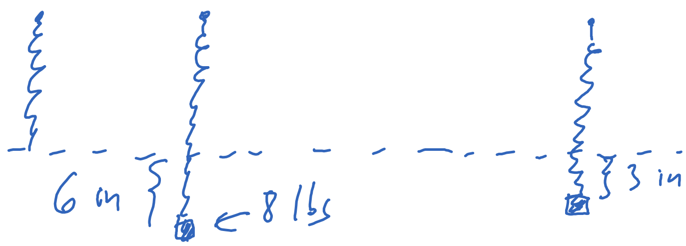
mention this first

$$\begin{cases} my'' + \gamma y' + ky = 0 \\ y = \begin{cases} c_1 e^{-r_1 t} + c_2 e^{-r_2 t} \rightsquigarrow \text{overdamped} : \gamma^2 > 4mk \\ e^{-\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t) \rightsquigarrow \text{underdamped} : \gamma^2 < 4mk \\ e^{-rt}(c_1 + c_2 t) \rightsquigarrow \text{critically damped} : \gamma^2 = 4mk \end{cases} \end{cases}$$

Interesting case: $\gamma = 0$.

$\omega^* = \omega_0 = \sqrt{\frac{k}{m}}$: the force frequency is the same as the natural frequency.

Ex



force $= 8\sin(8t)$

resistance $= \frac{1}{2}$ lbs when velocity $= 3$ in/s.

$k = \frac{\text{weight}}{\text{stretch}} = \frac{8}{0.5} = 16$

$m = \frac{\text{weight}}{g} = \frac{8}{32} = \frac{1}{4}$

$\gamma = \frac{\text{resistance}}{\text{velocity}} = \frac{1/2}{3/12} = 2$

$\frac{1}{4}y'' + 2y' + 16y = 8\sin(8t)$

$$\begin{cases} y'' + 8y' + 64y = 32\sin(8t) \\ y(0) = \frac{-3}{12} = -\frac{1}{4} \\ y'(0) = 0 \end{cases}$$

$y_c = e^{-4t}(c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t)$

$y_p = A\cos 8t + B\sin 8t \rightsquigarrow A = -\frac{1}{2},\ B = 0.$

$$y = \underbrace{-\frac{1}{2}\cos 8t}_{\text{Steady state sol}} + \underbrace{e^{-4t}\left(\frac{1}{4}\cos 4\sqrt{3}t + \frac{\sqrt{3}}{12}\sin 4\sqrt{3}t\right)}_{\text{transient sol}}$$

quasi-frequency

<u>question</u>: what frequency of the force should we choose to get resonance?

$F = 8\sin(\omega t)$.

Guess: $64 \times y_p = A\cos\omega t + B\sin\omega t \rightsquigarrow$ amplitude $= \sqrt{A^2+B^2}$

$8 \times y_p' = \omega B\cos\omega t - A\omega\sin\omega t$

$1 \times y_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

$$32\sin\omega t = (\underbrace{-A\omega^2 + 8\omega B + 64A}_{=0})\cos\omega t + (\underbrace{-B\omega^2 - 8A\omega + 64B}_{=32})\sin\omega t$$

$$B = \frac{A\omega^2 - 64A}{8\omega} = \frac{A(\omega^2-64)}{8\omega}$$

$$-8A\omega - B(\omega^2-64) = 32 \rightsquigarrow 8A\omega + \frac{A(\omega^2-64)^2}{8\omega} = -32$$

$$\rightsquigarrow A\left(8\omega + \frac{(\omega^2-64)^2}{8\omega}\right) = -32$$

$$A^2+B^2 = A^2\left(1+\left(\frac{B}{A}\right)^2\right) = 32^2\left(8\omega + \frac{(\omega^2-64)^2}{8\omega}\right)^{-2}\left(1+\frac{(\omega^2-64)^2}{(8\omega)^2}\right)$$

$$= 32^2(8\omega)^{-2}\left(1+\frac{(\omega^2-64)^2}{(8\omega)^2}\right)^{-1} = 32^2\left((8\omega)^2+(\omega^2-64)^2\right)^{-1}$$

$$\sqrt{A^2+B^2} = 32\frac{1}{\sqrt{(8\omega)^2+(\omega^2-64)^2}} \rightarrow \text{max when } \omega^2 = 32 \rightsquigarrow \underline{\underline{\omega = 4\sqrt{2}}}$$